Changes in the form of short gravity waves on long waves and tidal currents

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Short gravity waves, when superposed on much longer waves of the same type, have a tendency to become both shorter and steeper at the crests of the longer waves, and correspondingly longer and lower in the troughs. In the present paper, by taking into account the non-linear interactions between the two wave trains, the changes in wavelength and amplitude of the shorter wave train are rigorously calculated. The results differ in some essentials from previous estimates by Unna. The variation in energy of the short waves is shown to correspond to work done by the longer waves against the *radiation stress* of the short waves, which has previously been overlooked. The concept of the radiation stress is likely to be valuable in other problems.

1. Introduction

It is well known that when gravity waves of fairly short wavelength ride upon the surface of much longer waves such as ocean swell or tidal currents then the wavelength of the short waves is diminished at the crests of the long waves and increased in the troughs. The phenomenon was pointed out by Unna in a series of papers (1941, 1942, 1947). The relative shrinking of the short wavelength L' compared to its mean value L was expressed by Unna (1947) as

$$\frac{L'}{L} = 1 - a_2 k_2 \coth k_2 h \tag{1.1}$$

at the crests of the long waves, where a_2 denotes the amplitude and $2\pi/k_2$ the wavelength of the long waves; h denotes the total mean depth.

Besides this contraction of the wavelength on the long-wave crests, the amplitude of the short waves can be expected to be correspondingly increased. On intuitive grounds Unna (1947) suggested the formula

$$\frac{a'}{a_1} = 1 + a_2 k_2 \coth k_2 h, \tag{1.2}$$

where a_1 is the mean value of a'.

Being unconvinced by Unna's reasoning, we carried out a systematic evaluation of the wave motion by Stokes's method of approximation, as far as the second order. This method allows one to calculate rigorously the change in wavelength and amplitude arising from non-linear interactions between the two wave trains. The results are given in § 2 of the present paper. Equation (1.1) is verified, but in place of (1.2) we find

$$\frac{a'}{a_1} = 1 + a_2 k_2 (\tfrac{3}{4}\coth k_2 h + \tfrac{1}{4}\tanh k_2 h). \tag{1.3}$$

In deep water (when $k_2 h \to \infty$) both (1.2) and (1.3) tend to the same result

$$\frac{a'}{a_1} = 1 + a_2 k_2. \tag{1.4}$$

An interesting physical interpretation of (1.3) can be given. In § 3 of this paper it is shown that when a train of gravity waves of amplitude a ride upon a steady current U, the transfer of energy across any vertical plane normal to the motion is the sum of four terms

$$Ec_g + EU + S_x U + \tfrac{1}{2}\rho h U'^3, \tag{1.5}$$

where E denotes the mean energy density, c_g denotes the group velocity and U' is a modified stream velocity. S_x is defined below. The first two terms of (1.5) represent simply the bodily transport of energy by the group velocity c_g and by the stream velocity U. The last term in (1.5) represents the transport by the stream U' of its own kinetic energy. All these terms are to be expected. However, the third term $S_x U$ represents the work done by the current U against the *radiation stress* of the waves. S_x is given by

$$S_x = E\left(\frac{2c_g}{c} - \frac{1}{2}\right), \tag{1.6}$$

which for short waves reduces to $\frac{1}{2}E$. The quantity S_x is one component of a two-dimensional *stress tensor* defined in § 3. The presence of this term does not seem to have been pointed out previously.

If the short waves are riding not upon a uniform current but upon much longer waves, then the alternate contraction and expansion at the surface of the longer waves results in work being done against the radiation stress of the short waves. In § 5 it is shown that if this work is assumed to appear as additional energy in the short waves, then there must be a change in the amplitude of the short waves precisely by (1.3) This confirms the conclusions of § 2.

By the same method we are also able to calculate the change in the form of short waves riding on very long waves such as tidal currents. Setting $k_2 h \ll 1$ in equation (1.3) gives

$$\frac{a'}{a} \doteqdot 1 + \frac{3a_2}{4h}. \tag{1.7}$$

This, however, is valid only when the ratio of the wave frequencies σ_2/σ_1 is still small compared with $k_2 h$. If both σ_2/σ_1 and $k_2 h$ are small but of the same order of magnitude we find, on the crests of the long waves,

$$\frac{a'}{a} = 1 + a_2 k_2 \frac{3k_2 h - 2\sigma_2/\sigma_1}{(2k_2 h - \sigma_2/\sigma_1)^2}. \tag{1.8}$$

This reduces to (1.7) when $\sigma_2/\sigma_1 \ll k_2 h$.

The results of the present paper may be extended without difficulty to systems of waves crossing at an arbitrary angle, and to wavelengths short enough to be influenced by capillarity. In the latter case, however, viscosity probably plays a predominant role.

2. Determination of the wave profile

In this section we shall give a rigorous evaluation of the wave motion by the method of Stokes (1847) as far as the second approximation.

It is well known that in a real fluid the motion does not remain irrotational for long after it is generated from rest, and that a second-order vorticity ultimately penetrates the interior (Longuet-Higgins 1953). However, except in the boundary layers, which are very thin, the vorticity is quasi-steady and produces only second-order currents which, to the second approximation, are simply superposed upon the oscillatory motion. Since we shall be concerned only with the oscillatory part of the motion, it is therefore sufficient to assume the existence of a velocity potential ϕ; any steady second-order currents may be added afterwards.

Infinite depth

Take rectangular axes with the x-axis horizontal in the mean surface and the z-axis vertically upwards. Let $\mathbf{u}$, p, ρ, ζ denote the velocity, pressure, density and surface elevation respectively. Within the fluid we have the following relations

$$\left.\begin{aligned} \mathbf{u} &= \nabla\phi, \\ \nabla^2\phi &= 0, \\ \frac{p}{\rho}+gz+\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t} &= 0, \end{aligned}\right\} \tag{2.1}$$

the second equation being the equation of continuity and the third being Bernoulli's integral with the arbitrary function of t absorbed into ϕ. The boundary conditions are

$$\left.\begin{aligned} \left(\frac{p_0}{\rho}+gz+\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)_{z=\zeta} &= 0, \\ \left(\frac{\partial\zeta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}-\frac{\partial\phi}{\partial z}\right)_{z=\zeta} &= 0, \end{aligned}\right\} \tag{2.2}$$

and

$$\lim_{z\to-\infty} \nabla\phi = 0, \tag{2.3}$$

where p_0 denotes the pressure at the free surface (hereafter assumed to be zero). The surface conditions (2.2) may be replaced by conditions to be satisfied at $z = 0$ by assuming the left-hand sides to be expansible in a power series in z

$$\left.\begin{aligned} g\zeta+\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)_{z=0}+\zeta\left[\frac{\partial}{\partial z}\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)\right]_{z=0}+\ldots &= 0, \\ \frac{\partial\zeta}{\partial t}+\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}-\frac{\partial\phi}{\partial z}\right)_{z=0}+\zeta\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}-\frac{\partial\phi}{\partial z}\right)\right]_{z=0}+\ldots &= 0. \end{aligned}\right\} \tag{2.4}$$

Now let us assume expansions of the form

$$\left.\begin{aligned} \mathbf{u} &= \epsilon\mathbf{u}^{(1)}+\epsilon^2\mathbf{u}^{(2)}+\ldots,\\ \phi &= \epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\ldots,\\ \zeta &= \epsilon\zeta^{(1)}+\epsilon^2\zeta^{(2)}+\ldots,\\ \frac{p}{\rho}+gz &= \epsilon p^{(1)}+\epsilon^2 p^{(2)}+\ldots,\end{aligned}\right\} \tag{2.5}$$

where ϵ is a small quantity.* On substituting in equations (2.1), (2.3), (2.4), we have

$$\left.\begin{aligned} \mathbf{u}^{(1)} &= \nabla\phi^{(1)},\\ \frac{p^{(1)}}{\rho}+\frac{\partial\phi^{(1)}}{\partial t} &= 0,\end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned} \nabla^2\phi^{(1)} &= 0,\\ \lim_{z\to-\infty} \nabla\phi^{(1)} &= 0,\end{aligned}\right\} \tag{2.7}$$

and

$$\left.\begin{aligned} g\zeta^{(1)}+\left(\frac{\partial\phi^{(1)}}{\partial t}\right)_{z=0} &= 0,\\ \frac{\partial\zeta^{(1)}}{\partial t}-\left(\frac{\partial\phi^{(1)}}{\partial z}\right)_{z=0} &= 0.\end{aligned}\right\} \tag{2.8}$$

Elimination of $\zeta^{(1)}$ from the last two equations gives

$$\left(\frac{\partial^2\phi^{(1)}}{\partial t^2}+g\frac{\partial\phi^{(1)}}{\partial z}\right)_{z=0} = 0. \tag{2.9}$$

Equations (2.7) and (2.9) are equations for $\phi^{(1)}$ alone, while the remaining equations give $\mathbf{u}^{(1)}$, $p^{(1)}$ and $\zeta^{(1)}$ in terms of $\phi^{(1)}$.

As a solution of these equations we select the first-order motion corresponding to two progressive surface waves of wave-numbers k_1 and k_2; that is

$$\phi^{(1)} = A_1 e^{k_1 z}\cos(k_1 x-\sigma_1 t+\theta_1)+A_2 e^{k_2 z}\cos(k_2 x-\sigma_2 t+\theta_2), \tag{2.10}$$

where $A_1, A_2, \sigma_1, \sigma_2, k_1, k_2$ are constants and

$$\sigma_1^2 = gk_1, \quad \sigma_2^2 = gk_2. \tag{2.11}$$

The corresponding free surface is given by

$$\zeta^{(1)} = a_1\sin(k_1 x-\sigma_1 t+\theta_1)+a_2\sin(k_2 x+\sigma_2 t+\theta_2), \tag{2.12}$$

where

$$a_1 = -\frac{A_1 k_1}{\sigma_1}, \quad a_2 = -\frac{A_2 k_2}{\sigma_2}. \tag{2.13}$$

* ϵ is proportional roughly to the surface slope; here, however, ϵ will be used only as an ordering parameter.

Proceeding now to the second approximation, we have to satisfy

$$\left.\begin{aligned} &\mathbf{u}^{(2)} = \nabla\phi^{(2)}, \\ &\frac{p^{(2)}}{\rho} + \tfrac{1}{2}\mathbf{u}^{(1)2} + \frac{\partial\phi^{(2)}}{\partial t} = 0, \\ &\nabla^2\phi^{(2)} = 0, \\ &\lim_{z\to-\infty} \nabla\phi^{(2)} = 0, \\ &g\zeta^{(2)} + \left(\tfrac{1}{2}\mathbf{u}^{(1)2} + \frac{\partial\phi^{(2)}}{\partial t} + \zeta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial z\,\partial t}\right)_{z=0} = 0, \\ &\frac{\partial\zeta^{(2)}}{\partial t} + \left(\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial\zeta^{(1)}}{\partial x} - \frac{\partial\phi^{(2)}}{\partial z} + \zeta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial z^2}\right)_{z=0} = 0. \end{aligned}\right\} \tag{2.14}$$

Elimination of $\zeta^{(2)}$ from the last two equations gives

$$\left(\frac{\partial^2\phi^{(2)}}{\partial t^2} + g\frac{\partial\phi^{(2)}}{\partial z}\right)_{z=0} = -\left[\frac{\partial}{\partial t}(\mathbf{u}^{(1)2}) + \zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi^{(1)}}{\partial t^2} + g\frac{\partial\phi^{(1)}}{\partial z}\right)\right]_{z=0}. \tag{2.15}$$

On substituting the special solution (2.10) in the right-hand side, we see that the last group of terms vanishes identically, and we have

$$\begin{aligned} \frac{\partial^2\phi^{(2)}}{\partial t^2} + g\frac{\partial\phi^{(2)}}{\partial z} &= -\frac{\partial}{\partial t}(\mathbf{u}^{(1)2}) \\ &= 2A_1A_2k_1k_2(\sigma_1-\sigma_2)\cos\{(k_1-k_2)x-(\sigma_1-\sigma_2)t+(\theta_1-\theta_2)+\tfrac{1}{2}\pi\}. \end{aligned} \tag{2.16}$$

This and the above equations for $\phi^{(2)}$ are satisfied by

$$\phi^{(2)} = (A_1A_2k_1k_2/\sigma_2)\,e^{(k_1-k_2)z}\cos\{(k_1-k_2)x-(\sigma_1-\sigma_2)t+(\theta_1-\theta_2-\tfrac{1}{2}\pi)\}+Ct, \tag{2.17}$$

where C is an arbitrary constant to be determined by the condition that the origin is in the mean surface level. In fact $\zeta^{(2)}$ may be found from (2.14):

$$g\zeta^{(2)} = -\left(\frac{\partial\phi^{(2)}}{\partial t} + \tfrac{1}{2}\mathbf{u}^{(1)2} + \zeta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial z\,\partial t}\right)_{z=0}. \tag{2.18}$$

On making the substitutions and writing for short

$$k_1x-\sigma_1t+\theta_1 = \psi_1, \quad k_2x-\sigma_2t+\theta_2 = \psi_2, \tag{2.19}$$

we find

$$\zeta^{(2)} = -\tfrac{1}{2}a_1^2k_1\sin 2\psi_1 - \tfrac{1}{2}a_2^2k_2\sin 2\psi_2, \quad -a_1a_2(k_1\cos\psi_1\cos\psi_2 - k_2\sin\psi_1\sin\psi_2). \tag{2.20}$$

Thus if the small parameter ϵ is absorbed into a_1, a_2 by writing $\epsilon = 1$, we have

$$\begin{aligned} \zeta = {} & (a_1\sin\psi_1 - \tfrac{1}{2}a_1^2k_1\sin 2\psi_1) + (a_2\sin\psi_2 - \tfrac{1}{2}a_2^2k_2\sin 2\psi_2) \\ & -a_1a_2(k_1\cos\psi_1\cos\psi_2 - k_2\sin\psi_1\sin\psi_2) + \ldots. \end{aligned} \tag{2.21}$$

It is supposed that one of the waves is short compared with the other, say $k_1 \gg k_2$, and we wish to examine the influence of the second wave upon the first. For this purpose the terms in a_1^2, a_2, a_2^2 are irrelevant, and the remaining terms in (2.21) may be written

$$a_1\sin\psi_1(1+a_2k_2\sin\psi_2) - a_1\cos\psi_1(a_2k_1\cos\psi_2). \tag{2.22}$$

Now if P, Q are any small quantities (varying slowly compared to ψ_1), the expression

$$\zeta = a_1(1+P)\sin\psi_1 + a_1 Q\cos\psi_1 \tag{2.23}$$

represents a wave of slightly modified amplitude

$$a' = a_1(1+P), \tag{2.24}$$

and of slightly modified wave-number

$$k' = k_1\left(1+\frac{1}{k_1}\frac{\partial Q}{\partial x}\right). \tag{2.25}$$

Writing

$$P = a_2 k_2 \sin\psi_2, \quad Q = -a_2 k_1 \cos\psi_2, \tag{2.26}$$

we see that the amplitude of the small waves is increased by a factor

$$\frac{a'}{a_1} = 1+P = 1+a_2 k_2 \sin\psi_2, \tag{2.27}$$

and the wave-number is increased by the same factor; the wavelength is therefore correspondingly reduced. This factor varies between $(1+a_2k_2)$ on the crests of the long wave and $(1-a_2k_2)$ in the troughs.

Finite depth

We now suppose that the water is of uniform finite depth h and that $k_1 h$, $k_2 h$ are not necessarily large. The boundary condition at the bottom $(z=-h)$ is that the vertical velocity vanishes

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=-h} = 0, \tag{2.28}$$

and so

$$\left(\frac{\partial\phi^{(1)}}{\partial z}\right)_{z=-h} = \left(\frac{\partial\phi^{(2)}}{\partial z}\right)_{z=-h} = 0. \tag{2.29}$$

In order that the elevation of the free surface may be given by

$$\zeta^{(1)} = a_1\sin\psi_1 + a_2\sin\psi_2 \tag{2.30}$$

in the first approximation, we must have

$$\phi^{(1)} = -\frac{a_1\sigma_1}{k_1\sinh k_1 h}\cosh k_1(z+h)\cos\psi_1 - \frac{a_2\sigma_2}{k_2\sinh k_2 h}\cosh k_2(z+h)\cos\psi_2, \tag{2.31}$$

where

$$\sigma_1^2 = gk_1\tanh k_1 h, \quad \sigma_2^2 = gk_2\tanh k_2 h. \tag{2.32}$$

The evaluation of the second approximation now proceeds exactly as before. The algebra is somewhat longer, but may be simplified by omitting all terms except those involving the product $a_1 a_2$ in which alone we are interested. This being understood we have for the surface condition

$$\left(\frac{\partial^2\phi^{(2)}}{\partial t^2} + g\frac{\partial\phi^{(2)}}{\partial z}\right)_{z=0} = A\sin(\psi_1-\psi_2) + B\sin(\psi_1+\psi_2), \tag{2.33}$$

where

$$\left.\begin{aligned} A &= -\tfrac{1}{2}a_1a_2[2\sigma_1\sigma_2(\sigma_1-\sigma_2)(1+\alpha_1\alpha_2)+\sigma_1^3(\alpha_1^2-1)-\sigma_2^3(\alpha_2^2-1)],\\ B &= -\tfrac{1}{2}a_1a_2[2\sigma_1\sigma_2(\sigma_1+\sigma_2)(1-\alpha_1\alpha_2)-\sigma_1^3(\alpha_1^2-1)-\sigma_2^3(\alpha_2^2-1)],\end{aligned}\right\} \tag{2.34}$$

and where we have written

$$\alpha_1 = \coth k_1 h, \quad \alpha_2 = \coth k_2 h \tag{2.35}$$

for brevity. The solution of this equation satisfying Laplace's equation and equation (2.29) is

$$\phi^{(2)} = \frac{A \cosh (k_1 - k_2)(z+h) \sin (\psi_1 - \psi_2)]}{-(\sigma_1 - \sigma_2)^2 \cosh (k_1 - k_2)h + g(k_1 - k_2) \sinh (k_1 - k_2)h} + \frac{B \cosh (k_1 + k_2)(z+h) \sin (\psi_1 + \psi_2)}{-(\sigma_1 + \sigma_2)^2 \cosh (k_1 + k_2)h + g(k_1 + k_2) \sinh (k_1 + k_2)h}. \tag{2.36}$$

On substituting this expression in (2.18) and using the period equations (2.32), we find

$$g\zeta^{(2)} = \tfrac{1}{2} a_1 a_2 [C \cos (\psi_1 - \psi_2) - D \cos (\psi_1 + \psi_2)], \tag{2.37}$$

where

$$C = \frac{[2\sigma_1 \sigma_2 (\sigma_1 - \sigma_2)(1 + \alpha_1 \alpha_2) + \sigma_1^3 (\alpha_1^2 - 1) - \sigma_2^3 (\alpha_2^2 - 1)](\sigma_1 - \sigma_2)(\alpha_1 \alpha_2 - 1)}{\sigma_1^2 (\alpha_1^2 - 1) - 2\sigma_1 \sigma_2 (\alpha_1 \alpha_2 - 1) + \sigma_2^2 (\alpha_2^2 - 1)} + (\sigma_1^2 + \sigma_2^2) - \sigma_1 \sigma_2 (\alpha_1 \alpha_2 + 1), \tag{2.38}$$

and D is given by a similar expression with the signs of α_2, σ_2 reversed. A more convenient form for $\zeta^{(2)}$, equivalent to the above, is

$$\zeta^{(2)} = a_1 a_2 k_1 / \alpha_1 [E \cos \psi_1 \cos \psi_2 + F \sin \psi_1 \sin \psi_2], \tag{2.39}$$

where

$$E = \frac{\alpha_1 \alpha_2 [(\alpha_1^2 - 1)^2 - \lambda^2 (3\alpha_1^2 + 1)(\alpha_2^2 - 1) - \lambda^4 (3\alpha_2^2 + 1)(\alpha_1^2 - 1) + \lambda^6 (\alpha_2^2 - 1)^2] + 2\lambda^3 (\alpha_1^2 \alpha_2^2 - 1)(\alpha_1^2 + \alpha_2^2)}{[(\alpha_1^2 - 1) - 2\lambda \alpha_1 \alpha_2 + \lambda^2 (\alpha_2^2 - 1)]^2 - 4\lambda^2}, \tag{2.40}$$

$$F = \frac{-2\alpha_1 \alpha_2 [\lambda (\alpha_1^4 - 1) + \lambda^5 (\alpha_2^4 - 1)] + (\alpha_1^2 + \alpha_2^2 + 2\alpha_1^2 \alpha_2^2)[\lambda^2 (\alpha_1^2 - 1) + \lambda^4 (\alpha_2^2 - 1)]}{[(\alpha_1^2 - 1) - 2\lambda \alpha_1 \alpha_2 + \lambda^2 (\alpha_2^2 - 1)]^2 - 4\lambda^2}, \tag{2.41}$$

and where we have written

$$\sigma_2 / \sigma_1 = \lambda. \tag{2.42}$$

The quantities P, Q of equation (2.23) are now given by

$$\left. \begin{aligned} P &= (a_2 k_1 / \alpha_1) F \sin \psi_2, \\ Q &= (a_2 k_1 / \alpha_1) E \cos \psi_2. \end{aligned} \right\} \tag{2.43}$$

The case of deep water is easily retrieved from the above expressions by letting

$$\alpha_1 \to 1, \quad \alpha_2 \to 1 \tag{2.44}$$

(in that order, since $\sigma_1 > \sigma_2$). We then find

$$E \to -1, \quad F \to \lambda^2 = k_2 / k_1, \tag{2.45}$$

and the equations (2.43) for P and Q reduce to (2.26).

Let us now suppose that the shorter waves are effectively in deep water ($e^{-k_1 h}$ is negligible) but that the depth h is not necessarily great compared with the wavelength of the long waves. Under these conditions α_1 tends to 1 and $\lambda^2(\alpha_2^2-1)$ becomes a factor in both numerator and denominator of E and F. Hence

$$\left.\begin{aligned} E &= \frac{-2\alpha_2(2-\lambda\alpha_2)+2\lambda-\lambda^4\alpha_2(\alpha_2^2-1)}{(2-\lambda\alpha_2)^2-\lambda^2}, \\ F &= \frac{-2\alpha_2\lambda^3(1+\alpha_2^2)+\lambda^2(1+3\alpha_2^2)}{(2-\lambda\alpha_2)^2-\lambda^2}. \end{aligned}\right\} \quad (2.46)$$

Of particular interest to us is the case when λ is very small. Then

$$E = -\alpha_2, \quad F = \lambda^2\frac{1+3\alpha_2^2}{4}, \quad (2.47)$$

and so

$$\left.\begin{aligned} P &= a_2k_2\frac{1+3\alpha_2^2}{4\alpha_2}\sin\psi_2, \\ Q &= -a_2k_2\alpha_2\cos\psi_2. \end{aligned}\right\} \quad (2.48)$$

Hence the wave amplitude is increased by a factor

$$\frac{a'}{a_1} = 1 + a_2k_2(\tfrac{1}{4}\tanh k_2h + \tfrac{3}{4}\coth k_2h)\sin\psi_2 \quad (2.49)$$

and the wave-number is increased by a factor

$$\frac{k'}{k_1} = 1 + a_2k_2\coth k_2h\sin\psi_2. \quad (2.50)$$

This is always assuming that k_2h is not very small also.

The case when the longer waves are effectively in shallow water, that is

$$\tanh k_2h = \frac{1}{\alpha_1} \equiv \mu \ll 1 \quad (2.51)$$

may also be studied. Such a situation may occur, for example, with waves riding on a tidal current. But the small quantities λ, μ may be of the same order of magnitude. In a typical situation we might have waves of period $T_1 = 10$ seconds riding on a tidal stream of period $T_2 = 12{\cdot}4$ hours, in 50 fathoms of water. Then

$$\lambda = \frac{\sigma_2}{\sigma_1} = \frac{T_1}{T_2} = 2{\cdot}2\times10^{-4} \quad (2.52)$$

and

$$\mu \doteqdot k_2h = \frac{2\pi h}{T_2\sqrt{(gh)}} = 4{\cdot}4\times10^{-4}. \quad (2.53)$$

Retaining the terms of lowest order in both λ and μ, we find from (2.46)

$$\left.\begin{aligned} E &= -\frac{2}{2\mu-\lambda}+\frac{\lambda^4}{\mu(2\mu-\lambda)^2}, \\ F &= \frac{\lambda^2(3\mu-2\lambda)}{\mu(2\mu-\lambda)^2}, \end{aligned}\right\} \quad (2.54)$$

and so from (2.43)

$$\left.\begin{aligned} P &= a_2k_2\frac{3\mu-2\lambda}{(2\mu-\lambda)^2}\sin\psi_2, \\ Q &= -a_2k_1\left\{\frac{2}{2\mu-\lambda}-\frac{\lambda^4}{\mu(2\mu-\lambda)^2}\right\}\cos\psi_2. \end{aligned}\right\} \quad (2.55)$$

The changes in wave amplitude and wave-number are therefore given by

$$\frac{a'}{a_1} = 1 + a_2 k_2 \frac{3\mu - 2\lambda}{(2\mu-\lambda)^2} \sin\psi_2, \tag{2.56}$$

and

$$\frac{k'}{k_1} = 1 + a_2 k_2 \left\{\frac{2}{2\mu-\lambda} - \frac{\lambda^4}{\mu(2\mu-\lambda)^2}\right\} \sin\psi_2. \tag{2.57}$$

When λ/μ is small these equations reduce to

$$\frac{a'}{a_1} = 1 + \frac{3a_2}{4h} \sin\psi_2, \tag{2.58}$$

and

$$\frac{k'}{k} = 1 + \frac{a_2}{h} \sin\psi_2, \tag{2.59}$$

respectively.

The above results for shallow water may also be deduced directly starting from a velocity potential

$$\phi^{(1)} = -\frac{a_1 \sigma_1}{k_1} e^{k_1 z} \cos\psi_1 - \frac{a_2 \sigma_2}{k_2^2 h} [1 + \tfrac{1}{2} k_2^2 (z+h)^2] \cos\psi_2. \tag{2.60}$$

The only other special cases of interest are when the longer waves are shallow-water waves and the shorter waves are either shallow-water or deep-water waves. However, the results are all contained in equation (2.41), and the appropriate simplifications may be left to the reader.

Standing waves

So far we have considered only waves of progressive type riding on longer waves, also of progressive type, travelling in the same direction.

It is evident, however, that when λ is very small ($\lambda \ll \mu$) the expressions for the shortening or the steepening of the waves are unaffected if λ is reversed in sign, that is if the direction of one of the waves is reversed. Hence the shortening and steepening of the waves are the same whether the second system of waves is travelling in the same or opposite direction to the first.

Further, the interaction terms, on which these effects depend, are evidently linear in the two wave amplitudes a_1, a_2 separately. It follows that, if two of the longer waves are superposed to give a standing wave, and if the short progressive waves ride on top of these, the relative shortening and steepening will be similar. More precisely if

$$\begin{aligned} \zeta^{(1)} &= a_1 \sin(k_1 x - \sigma_1 t + \theta_1) + a_2 \sin(k_2 x - \sigma_2 t + \theta_2) + a_2 \sin(k_2 x + \sigma_2 t + \theta_2) \\ &= a_1 \sin(k_1 x - \sigma_1 t + \theta_1) + 2a_2 \sin k_2 x \cos(\sigma_2 t + \theta_2), \end{aligned} \tag{2.61}$$

then on the crests of the longer waves the amplitude of the shorter waves is found to increase by a factor

$$1 + a_2 k_2 (\tfrac{1}{2} \tanh k_2 h + \tfrac{3}{2} \coth kh), \tag{2.62}$$

and the wavelength is diminished by a factor

$$1 + 2a_2 k_2 \coth k_2 h. \tag{2.63}$$

If a_2 is written for $2a_2$ these formulae are similar to (2.49) and (2.50).

Similarly, if two *short* waves are added to produce a short standing wave, then by the linearity of the interaction terms it follows that the changes in amplitude and wavelength of the combined wave are given by identical expressions, both when the longer waves are progressive and when they are standing waves.

On the other hand, in shallow water when λ is not much less than μ, the change of form of short progressive waves depends upon their direction relative to the longer waves. Hence different formulae for standing waves result, which may be deduced without difficulty from equations (2.56) and (2.57).

3. The radiation stresses

In order to interpret physically the conclusions of § 2, we first consider from a general point of view the transfer of energy by surface waves on a steady, uniform current.

In a non-viscous fluid, the rate of transfer of energy across a surface fixed in space is given by

$$R = \iint_S (p + \tfrac{1}{2}\rho \mathbf{u}^2 + \rho g z)\, \mathbf{u} . \mathbf{n}\, dS, \tag{3.1}$$

where $\mathbf{n}$ denotes the unit normal to the surface, and z is measured vertically upwards. Hence the mean rate of transfer across a vertical plane $x = \text{const.}$, per unit distance in the y-direction, is

$$R_x = \overline{\int_{-h}^{\zeta} (p + \tfrac{1}{2}\rho \mathbf{u}^2 + \rho g z)\, u\, dz}, \tag{3.2}$$

where $z = \zeta(t)$ denotes the free surface, and the mean value with respect to time, indicated by a bar, is taken *after* performing the integration. We now express the velocity as the sum of two parts

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \tag{3.3}$$

where $\mathbf{U} = (U, 0, 0)$ denotes the mean stream velocity and $\mathbf{u}'$ is the additional velocity due to the wave motion. It may be assumed that the mean value of $\mathbf{u}'$ at any point in the *interior* is zero

$$\overline{\mathbf{u}'} = 0 \tag{3.4}$$

and further that U is independent of z.* On substituting (3.3) into (3.2) and taking mean values, we have identically

$$R_x = R_0 + R_1 + R_2 + R_3, \tag{3.5}$$

where

$$\left.\begin{aligned} R_0 &= \overline{\int_{-h}^{\zeta} (p + \tfrac{1}{2}\rho \mathbf{u}'^2 + \rho g z)\, u'\, dz}, \\ R_1 &= \overline{\int_{-h}^{\zeta} (p + \tfrac{1}{2}\rho \mathbf{u}'^2 + \rho g z + \rho u'^2)\, dz}\; U, \\ R_2 &= \overline{\int_{-h}^{\zeta} \tfrac{3}{2}\rho u'\, dz}\; U^2, \\ R_3 &= \overline{\int_{-h}^{\zeta} \tfrac{1}{2}\rho\, dz}\; U^3. \end{aligned}\right\} \tag{3.6}$$

* These assumptions taken together are valid only for irrotational flow; vorticity may be taken into account by supposing U to depend upon z.

Let us consider these terms separately. The first term R_0 is simply equal to the energy transfer by the waves in the absence of a steady stream. Adapting the notation of § 2, we have

$$\left.\begin{aligned} \zeta &= a\cos(kx-\sigma t+\theta)+O(a^2k), \\ \phi &= \frac{a\sigma}{k\sinh kh}\cosh k(z+h)\sin(kx-\sigma t+\theta)+O(a^2\sigma), \end{aligned}\right\} \tag{3.7}$$

where
$$\sigma^2 = gk\tanh kh, \quad \sigma/k = c. \tag{3.8}$$

Hence it is easily found that, to second order,

$$R_0 = \tfrac{1}{4}\rho g a^2 c\left(1+\frac{2kh}{\sinh 2kh}\right) = Ec_g, \tag{3.9}$$

where
$$E = \tfrac{1}{2}\rho g a^2 \tag{3.10}$$

denotes the mean energy density per unit horizontal area and

$$c_g = \frac{d\sigma}{dk} = \tfrac{1}{2}c\left(1+\frac{2kh}{\sinh 2kh}\right) \tag{3.11}$$

denotes the group velocity (cf. Lamb 1932, § 237).

The second term in (3.6) may be separated into two parts

$$R_1 = R_{11}+R_{12}, \tag{3.12}$$

where
$$\left.\begin{aligned} R_{11} &= \overline{\int_{-h}^{\zeta}(\tfrac{1}{2}\rho\mathbf{u}'^2+\rho g z)\,dz}\;U+\tfrac{1}{2}\rho g h^2 U = EU, \\ R_{12} &= \overline{\int_{-h}^{\zeta}(p+\rho u'^2)\,dz}\;U-\tfrac{1}{2}\rho g h^2 U = S_x U. \end{aligned}\right\} \tag{3.13}$$

The term R_{11} is self-explanatory; it is the bodily transport of kinetic and gravitational energy by the mean velocity U. The term R_{12} is more interesting and its presence does not seem to have been previously noticed. It represents the work done by the mean velocity U against the *radiation stress* defined by

$$S_x = \overline{\int_{-h}^{\zeta}(p+\rho u'^2)\,dz}-\tfrac{1}{2}\rho g h^2. \tag{3.14}$$

To interpret this expression we divide it again into two parts: first take the integral with respect to z up to a fixed point, say the mean level $z = 0$. (If $\zeta < 0$ then p and $\mathbf{u}$ may be extended analytically.) Thus we have

$$T_x = \int_{-h}^{0}\overline{(p+\rho u'^2)\,dz}-\tfrac{1}{2}\rho g h^2, \tag{3.15}$$

say. In this expression the quantity $\rho\overline{u'^2}$ represents the well-known Reynolds stress, which arises because the excess velocity u' transfers horizontal momentum $\rho u'$ at a rate $\rho u'^2$; even when u' is negative the contribution to the Reynolds stress is positive.

To obtain S_x we have only to add to T_x the quantity

$$Z_x = \overline{\int_{0}^{\zeta}(p+\rho u'^2)\,dz} \tag{3.16}$$

(the integral being interpreted in the usual way when $\zeta < 0$). In this expression the term $\rho u'^2$ contributes only a small quantity of the third order. The remaining term p gives a positive contribution to Z_x since when ζ is positive (the surface is above the mean level) so also is the pressure, and when ζ is negative so also is p; in fact near the mean level p is given almost by the hydrostatic pressure term $\rho g(\zeta - z)$; hence

$$Z_x \doteqdot \overline{\int_0^{\zeta} \rho g(\zeta - z)\, dz} = \tfrac{1}{2}\rho g\overline{\zeta^2}. \tag{3.17}$$

It will be seen that this term arises essentially from the deformation of the free surface.

On the other hand, to evaluate T_x we must express $\bar{p}$ to the second order in the wave amplitude a: assuming $\bar{\zeta} = 0$, we find

$$\bar{p} = -\rho g z - \frac{\rho g a^2 k}{\sinh 2kh} \cosh^2 k(z+h). \tag{3.18}$$

It will be seen that the second term on the right is negative, so that the mean pressure at a point is actually reduced by the presence of the waves. On substituting in (3.15) and (3.17) we have, to order a^2,

$$\left.\begin{aligned} T_x &= \tfrac{1}{2}\rho g a^2 \frac{2kh}{\sinh 2kh}, \\ Z_x &= \tfrac{1}{4}\rho g a^2. \end{aligned}\right\} \tag{3.19}$$

Combining these, we have

$$S_x = \tfrac{1}{2}\rho g a^2 \left(\frac{2kh}{\sinh 2kh} + \frac{1}{2}\right) = E\left(\frac{2c_g}{c} - \frac{1}{2}\right). \tag{3.20}$$

Thus S_x is an additional stress, due to the wave motion, per unit length across a plane normal to the direction of wave propagation. It is composed of the integrated Reynolds stress, plus the stress due to the correlation between surface elevation and pressure, less the effect of the reduction in the average pressure in the body of the fluid due to the presence of the waves. Altogether

$$R_1 = (E + S_x)\, U = E\left(\frac{2c_g}{c} + \frac{1}{2}\right) U. \tag{3.21}$$

The last two terms in (3.6) are easily evaluated. Since $\overline{u'}$ vanishes everywhere in the interior of the fluid, and $\bar{\zeta} = 0$, we have

$$R_2 = \tfrac{3}{2} E U^2 / c, \quad R_3 = \tfrac{1}{2}\rho h U^3. \tag{3.22}$$

But since the motion is irrotational there is, owing to the mass-transport velocity, a net momentum E/c in the direction of wave propagation (Stokes 1847), that is, a mean velocity $E/c\rho h$. Writing

$$U + E/c\rho h = U' \tag{3.23}$$

and substituting in (3.22) we have, to the present order of approximation,

$$R_2 + R_3 = \tfrac{1}{2}\rho h U'^3 \tag{3.24}$$

which represents the transport of the kinetic energy of the current by itself.

Collecting together the various terms, we find

$$R_x = Ec_g + EU + S_x U + \tfrac{1}{2}\rho h U'^3, \tag{3.25}$$

where S_x and U' are given by (3.20) and (3.23).

In an exactly similar way we may calculate the flow of energy in the y-direction in the presence of a steady transverse current $\mathbf{U} = (0, V, 0)$. This is given by the integral

$$R_y = \overline{\int_{-h}^{\zeta} (p + \tfrac{1}{2}\rho \mathbf{u}'^2 + \rho g z + V^2)\, V dz}, \tag{3.26}$$

which is easily found to be

$$R_y = EV + S_y V + \tfrac{1}{2}\rho h V^3, \tag{3.27}$$

where
$$S_y = \tfrac{1}{4}\rho g a^2 \frac{2kh}{\sinh 2kh} = E\left(\frac{c_g}{c} - \frac{1}{2}\right). \tag{3.28}$$

In the general case of a mean stream velocity $\mathbf{U} = (U, V, 0)$ the transfer of energy across a vertical plane in direction $\mathbf{n} = (l, m, 0)$ is

$$\bar{R} = \overline{\int_{-h}^{\zeta} (p + \tfrac{1}{2}\rho \mathbf{u}'^2 + \rho g z + \rho \mathbf{u}'.\mathbf{U} + \tfrac{1}{2}\rho \mathbf{U}^2)(\mathbf{u}' + \mathbf{U}).\mathbf{n}\, dz}, \tag{3.29}$$

which by exactly similar analysis is found to be

$$\bar{R} = E\mathbf{c}_g.\mathbf{n} + E\mathbf{U}.\mathbf{n} + \mathbf{U}.\mathbf{S}.\mathbf{n} + \tfrac{1}{2}\rho h \mathbf{U}'^2(\mathbf{U}'.\mathbf{n}), \tag{3.30}$$

where
$$\mathbf{c}_g = (c_g, 0, 0) \tag{3.31}$$

denotes the vector group velocity,

$$\mathbf{U}' = (U + E/\rho c h,\ V, 0) \tag{3.32}$$

denotes the modified stream velocity, and where $\mathbf{S}$ denotes the tensor

$$\mathbf{S} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix}. \tag{3.33}$$

$\mathbf{S}$ may be called the *stress tensor* of the wave motion. In full it is

$$\mathbf{S} = \begin{pmatrix} E\left(\frac{2c_g}{c} - \frac{1}{2}\right) & 0 \\ 0 & E\left(\frac{c_g}{c} - \frac{1}{2}\right) \end{pmatrix}. \tag{3.34}$$

In very deep water ($c_g = \tfrac{1}{2}c$) it becomes

$$\mathbf{S} = \begin{pmatrix} \tfrac{1}{2}E & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.35}$$

and in shallow water ($c_g = c$) it becomes

$$\mathbf{S} = \begin{pmatrix} \tfrac{3}{2}E & 0 \\ 0 & \tfrac{1}{2}E \end{pmatrix}. \tag{3.36}$$

It is interesting to note that there is also a transport of energy corresponding to a vertical component of velocity W across the horizontal plane z = constant. In fact the mean energy transport per unit horizontal area is

$$\overline{(p + \tfrac{1}{2}\rho \mathbf{u}'^2 + \rho g z + \rho w' W + \tfrac{1}{2}\rho W^2)(w' + W)}. \tag{3.37}$$

The terms independent of W together vanish identically (there is no upwards transport of energy in an ordinary surface wave). The terms proportional to W are

$$(p+\tfrac{1}{2}\rho\mathbf{u}'^2+\rho g z+\rho w'^2)\,W = \frac{\rho g a^2 k}{2\sinh 2kh}\cosh 2k(z+h)\,W. \tag{3.38}$$

In deep water ($kh \gg 1$) this becomes

$$\tfrac{1}{2}W\rho g a^2 k\, e^{-2kz}, \tag{3.39}$$

which is negligible below about half a wavelength.

4. The relation between wave amplitude and energy in an accelerated wave

In the preceding section we have calculated the transfer of energy horizontally when surface waves are superposed upon a steady, uniform current. We propose in the following to investigate the case when the surface waves ride not upon a steady current but upon a much longer wave, as in § 2, that is to say in place of the steady current U of § 3 we have instead the orbital velocity of the long waves. (This latter velocity is however supposed small compared with the phase velocity of the short waves.)

If the wavelength of the longer waves is sufficiently great compared with that of the shorter waves, then it is permissible to regard the orbital velocity U as being approximately constant and uniform over a period and wavelength, respectively, of the shorter waves. To a certain extent therefore we may make use of the formulae of § 3. However, a significant factor is introduced by the presence of a vertical acceleration in the longer waves; this alters the relation between the amplitude and the energy of the short waves, as will now be shown.

We shall consider from a general point of view the relation between the potential and the kinetic energy of a system undergoing vertical movements.

The discussion of energy relations in frames of reference not moving with constant acceleration leads generally to complications. Therefore we shall agree from the start to refer all energies to a stationary frame of reference.

In the stationary frame of reference, a progressive wave train of amplitude a' will have a gravitational potential energy

$$\text{P.E.} = \tfrac{1}{4}\rho g a'^2 \tag{4.1}$$

per unit horizontal distance (apart from terms independent of the wave amplitude a' and terms of higher order than the second).

Consider on the other hand the kinetic energy, measured in the same frame of reference. A very general theorem in dynamics states that the kinetic energy of a system of particles of mass m_i and velocity $\mathbf{v}_i$ is given by

$$\text{K.E.} = \tfrac{1}{2}M\mathbf{V}^2+\sum_i \tfrac{1}{2}m_i(\mathbf{v}_i-\mathbf{V})^2, \tag{4.2}$$

where M is the total mass and $\mathbf{V}$ the velocity of the centre of mass. Now the vertical co-ordinate Z of the centre of mass of a wave train differs from the vertical co-ordinate Z_S of the mean free surface by an amount

$$Z-Z_S = \tfrac{1}{4}\rho a'^2/M \quad + \text{constant} \tag{4.3}$$

(neglecting terms of higher order). Hence the vertical velocity of the centre of mass differs from that of the free surface by an amount

$$\mathbf{V}-\mathbf{V}_S = \left[0, 0, \frac{\partial}{\partial t}(\tfrac{1}{4}\rho a'^2/M)\right]. \tag{4.4}$$

The first term on the right of (4.2) can therefore be written

$$\tfrac{1}{2}M\mathbf{V}_S^2 + W_S\frac{\partial}{\partial t}(\tfrac{1}{4}\rho a'^2). \tag{4.5}$$

The last term in (4.2) represents simply the kinetic energy calculated with reference to a frame moving (nearly) with the free surface and is therefore given by

$$\tfrac{1}{4}\rho a'^2\sigma'^2/k', \tag{4.6}$$

where $2\pi/\sigma'$ and $2\pi/k'$ are the period and wavelength of the waves in the moving frame. But since this frame is accelerated these are related by the equations

$$\sigma'^2 = g'k', \tag{4.7}$$

where g' is the apparent value* of gravity

$$g' = g+\frac{\partial W_S}{\partial t}. \tag{4.8}$$

Altogether then we have

$$\text{K.E.} = \tfrac{1}{2}M\mathbf{V}_S^2 + \tfrac{1}{4}\rho g a'^2 + \frac{\partial}{\partial t}(\tfrac{1}{4}\rho a'^2 W_S). \tag{4.9}$$

The total wave energy E' may be defined as those parts of the kinetic-plus-potential energy which depend on the wave amplitude only, i.e.

$$E' = \tfrac{1}{2}\rho g a'^2 + \frac{\partial}{\partial t}(\tfrac{1}{4}\rho a'^2 W_S). \tag{4.10}$$

When $\partial a'^2/\partial t$ and W_S are both small quantities this expression becomes

$$E' = \tfrac{1}{2}\rho g a'^2\left(1+\frac{1}{2g}\frac{\partial W_S}{\partial t}\right). \tag{4.11}$$

5. A physical interpretation of the results of §2

In the situation described in §2 we may regard the shorter waves as being superposed upon the longer waves, whose orbital velocity near the free surface has the components

$$\left.\begin{aligned} U &= a_2\sigma_2\coth k_2h\,\sin\psi_2,\\ W &= -a_2\sigma_2\cos\psi_2.\end{aligned}\right\} \tag{5.1}$$

Consider first the changes in wavelength of the shorter waves. We make the physical assumption that *the wavelength of the short waves expands in proportion to the stretching of the surface by the long waves.*

* It is assumed that $(1/g')\,\partial g'/\partial t$ is small compared with σ'.

Now two particles in the surface which initially are separated by a distance dx have a small relative velocity $(\partial U/\partial x)\,dx$. The separation of these particles after time t is therefore given by

$$dx + \int_0^t \frac{\partial U}{\partial x}\,dx\,dt = dx\left[1 + \int_0^t \frac{\partial U}{\partial x}\,dt\right], \tag{5.2}$$

where to a first approximation $\partial U/\partial x$ may be evaluated at the original position x. The relative stretching of the surface is therefore given by

$$1 + \int_0^t \frac{\partial U}{\partial x}\,dt = 1 - a_2 k_2 \coth k_2 h \sin \psi_2. \tag{5.3}$$

The relative increase in wave-number of the short waves is the reciprocal of this expression, or

$$1 + a_2 k_2 \coth k_2 h \sin \psi_2, \tag{5.4}$$

in agreement with (2.50).

Now to account for the change in the wave amplitude we shall make the following assumptions.

(*a*) *The energy density of the short waves is given by* (4.11) (despite the distortion caused by stretching of the surface).

(*b*) *The rate of transfer of short-wave energy is given by*

$$E'(c_g + U) + S_x U \tag{5.5}$$

as in § 3.

(*c*) *The short-wave energy is conserved* (and in particular that work done against the radiation stress appears as short-wave energy).

With these assumptions the equation for the budget of short-wave energy becomes

$$\frac{\partial E'}{\partial t} = -\frac{\partial}{\partial x}[E'(c_g + U) + S_x U]. \tag{5.6}$$

To the order of approximation with which we are concerned we may take on the right-hand side of (5.6)

$$E' = \text{const.} = \tfrac{1}{2}\rho g a_1^2 = E_1, \tag{5.7}$$

and similarly $S_x = \text{const.}$, so that (5.6) reduces to

$$\frac{\partial E'}{\partial t} = -E_1 \frac{\partial}{\partial x}(c_g + U) - S_x \frac{\partial U}{\partial x}. \tag{5.8}$$

The physical interpretation of this last equation is that the rate of change of the short-wave content between x and $x + dx$ is determined by the divergence of the energy transport due to the group velocity c_g and the ambient flow U, plus the rate at which the convergence of the ambient flow, $(\partial U/\partial x)$, does work against the radiation stress S_x. Our assumption is that in this case the work done against the radiation stress appears as additional wave energy (although it is not possible to assert that such would be the case in other circumstances).

The term $S_x \partial U/\partial x$ appearing in (5.8) is closely analogous to the term $\overline{u_i u_j}\,\partial U_1/\partial x_j$ which appears in the equations for turbulent energy and the term $p\nabla . \mathbf{V}$ which occurs in the energy expression for turbulent flows.

Now, on relacing S_x by (3.19) and E by E_1, we have

$$\frac{\partial E'}{\partial t} = -E_1 \frac{\partial}{\partial x}\left[(c_g+U)+\left(\frac{2c_g}{c_1}-\frac{1}{2}\right)U\right], \tag{5.9}$$

or, since $\partial c_g/\partial x \ll \partial U/\partial x$,

$$\frac{\partial E'}{\partial t} = -E_1\left(\frac{2c_g}{c_1}+\frac{1}{2}\right)\frac{\partial U}{\partial x}. \tag{5.10}$$

Since U represents a progressive wave motion, the operation $\partial/\partial x$ may be replaced by $-(1/c_2)\,\partial/\partial t$, giving

$$\frac{\partial E'}{\partial t} = E_1\left(\frac{2c_g}{c_1}+\frac{1}{2}\right)\frac{1}{c_2}\frac{\partial U}{\partial t}. \tag{5.11}$$

Integration with respect to t (from an instant when the surface crosses the mean level and $U = 0$, $E = E_1$) gives

$$E' - E_1 = E_1\left(\frac{2c_g}{c}+\frac{1}{2}\right)\frac{U}{c_2}, \tag{5.12}$$

or

$$\frac{E'}{E_1} = 1+\left(\frac{2c_g}{c_1}+\frac{1}{2}\right)\frac{U}{c_2}. \tag{5.13}$$

Substituting for E' from (4.11), we obtain

$$\frac{a'^2}{a_1^2} = 1+\left(\frac{2c_g}{c_1}+\frac{1}{2}\right)\frac{U}{c_2}-\frac{1}{2g}\frac{\partial W}{\partial t}, \tag{5.14}$$

and so, since U and W are both of order $a_2\sigma_2$,

$$\frac{a'}{a_1} = 1+\left(\frac{c_g}{c_1}+\frac{1}{4}\right)\frac{U}{c_2}-\frac{1}{4g}\frac{\partial W}{\partial t}. \tag{5.15}$$

In the case when the shorter waves are in deep water, $c_g/c_1 = \frac{1}{2}$ and hence

$$\frac{a'}{a_1} = 1+\frac{3}{4}\frac{U}{c_2}-\frac{1}{4g}\frac{\partial W}{\partial t}. \tag{5.16}$$

Since, from (5.1) and (2.32),

$$\left.\begin{aligned}\frac{U}{c_2} &= a_2 k_2 \coth k_2 h \sin\psi_2,\\ -\frac{1}{g}\frac{\partial W}{\partial t} &= \frac{a_2\sigma_2^2}{g}\sin\psi_2 = a_2 k_2 \tanh k_2 h \sin\psi_2,\end{aligned}\right\} \tag{5.17}$$

equation (5.16) is equivalent to (2.49). Thus we have verified both equations (2.49) and (2.50) by alternative reasoning.

It will be seen that in shallow water, when the term $(1/g)\,\partial W/\partial t$ proportional to the vertical acceleration, is negligible, we have from (5.15) and (5.16)

$$\frac{a'}{a_1} = 1+\left(\frac{c_g}{c}+\frac{1}{4}\right)\frac{U}{c_2} \tag{5.18}$$

and

$$\frac{a'}{a_1} = 1+\frac{3}{4}\frac{U}{c_2}, \tag{5.19}$$

respectively, the last equation being equivalent to (2.58).

The derivation of (5.16), (5.18) and (5.19) does not depend upon the sinusoidal character of the longer waves but only upon their being progressive. These formulae can therefore be expected to remain valid for short waves riding on cnoidal or solitary waves, or any other kind of progressive disturbance, provided it is sufficiently long.

Equation (5.16) can be further generalized to any disturbance consisting of the sum of a number of wave motions in which the velocities $c_2^{(i)}$ may be positive or negative. Thus we have

$$\frac{a'}{a_1} = 1 + (\Sigma^{(+)} - \Sigma^{(-)}) \left(\frac{3}{4}\frac{U^{(i)}}{c_2^{(i)}} - \frac{1}{4g}\frac{\partial W^{(i)}}{\partial t}\right), \tag{5.20}$$

where $\Sigma^{(+)}$ denotes the sum over all values of i for which $c_2^{(i)}$ is positive, and $\Sigma^{(-)}$ the corresponding sum for $c_2^{(i)}$ negative. In shallow water, when $c_2^{(i)} = \pm\sqrt{(gh)}$ we have

$$\frac{a'}{a_1} = 1 + \frac{3}{4\sqrt{(gh)}}(\Sigma^{(+)}U^{(i)} - \Sigma^{(-)}U^{(i)}). \tag{5.21}$$

It should be noted that the present method is not capable of yielding in a simple way the more refined formulae (2.56) and (2.57) which are applicable when the ratio of the wave frequencies is no longer small compared with $k_2 h$. For deriving these, the longer but more rigorous method of § 2 is to be preferred.

6. On a result of Unna

As mentioned in § 1, a formula for the change in amplitude essentially different from that which we have found was suggested by Unna (1941, 1947); his result is stated in equation (1.2).

Unna apparently did not work out the wave interactions exactly but relied on a physical argument. His reasoning differs from ours in two respects. First, he neglects entirely the work done by the longer waves against the radiation stress S_x, which we have taken into account. Secondly, he calculates the potential energy of the waves in the accelerated frame of reference, replacing g by $g + \partial W_S/\partial t$ in equation (4.1). He then assumes that kinetic and potential energy are conserved in the *accelerated* system.

It is not difficult to show that the kinetic-plus-potential energy is not generally conserved in an accelerated frame of reference, even when the acceleration is slow compared to the natural period of oscillation of the system. As examples we may quote a simple pendulum hinged at a point which is accelerated vertically, or the oscillations of water in a U-tube likewise accelerated.

The argument from conservation of energy therefore fails unless it is applied in a fixed or inertial frame of reference, as in §§ 4 and 5. If an accelerated frame of reference is used it must be supposed that there is some kind of interaction between the dynamical system and the accelerating forces.

In the case of deep water ($k_2 h \gg 1$) it happens that Unna's two mistakes—neglect of the radiation stress and assumption of energy conservation in the accelerated system—exactly cancel. But that they do not generally cancel is shown by the difference between equations (1.2) and (1.3).

7. Conclusions

The change in wavelength of short waves on the crests of longer waves can be interpreted as being due simply to the contraction of the particles in the longer wave.

However, to account for the increase in the amplitude of the short waves it is necessary to allow for the work done by the longer waves against the *radiation stress* of the short waves. This work is converted into short-wave energy, and produces a steepening of the short waves beyond what was previously expected.

The radiation stress is likely to play an important part in other situations, for example in waves riding on steady but non-uniform currents. Without close examination it cannot be assumed that work done against the radiation stress must necessarily appear as additional wave energy. But we have shown that in the present situation at least this assumption proves correct.

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